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ON LIE'S APPROACH TO THE STUDY OF TRANSLATION MANIFOLDS

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1. Introduction

In this paper, we will study a question originally posed by Sophus Lie in [4]. After Lie completed his work on double translation surfaces, he began to consider several different kinds of generalizations of this special class of surfaces. To understand the particular types of manifolds he considered, it should be kept in mind that Lie interpreted his theorem showing that every nondevelopable double translation surface is a piece of the theta-divisor in the Jacobian of an algebraic curve of genus three as a characterization of the abelian integrals on the curve by the functional equations they satisfy as a result of Abel's theorem. Therefore, in addition to studying the higher-dimensional analogs of double translation surfaces, Lie also proposed the problem of determining if the abelian integrals on algebraic curves of a given genus may be characterized by other, more general, functional equations.

As a first step in a program left incomplete at his death, Lie undertook the study of analytic hypersurfaces $S \subset \mathbf{C}^4$ with two different parametrizations of the form

$$x_i = \alpha_i(t_1) + A_i(t_2, t_3) = \beta_i(u_1) + B_i(u_2, u_3). \tag{1}$$

Geometrically, the existence of parametrizations of this form implies that S may be swept out in two different ways by translating a curve along a two-dimensional surface. This condition is a natural generalization of the definition of double translation three-folds, a class of manifolds which Lie had studied previously in [3] (see also [5] and [7]). In [4], Lie referred to such hypersurfaces as "Translationsmannigfaltigkeiten zweiter Art." We will call them generalized double translation manifolds instead.

The existence of two parametrizations as in (1) is *apparently* a weaker hypothesis than the assumption that S is a double translation manifold, with

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two distinct parametrizations

$$(2) \qquad x_i = \alpha_{1i}(t_1) + \alpha_{2i}(t_2) + \alpha_{3i}(t_3) = \beta_{1i}(u_1) + \beta_{2i}(u_2) + \beta_{3i}(u_3).$$

However, Lie apparently believed and we will show that in fact, if S is not developable (that is, the Gauss mapping on S is *not degenerate* in the sense of [2]) and the generating curves and surfaces of S satisfy some general position hypotheses to be made explicit later, then the existence of two parametrizations of the form (1) *implies* the existence of two parametrizations of the form (2). Indeed, the surfaces

$$x_i = A_i(t_2, t_3) \quad \text{and} \quad x_i = B_i(u_2, u_3)$$

in S must themselves be *translation surfaces*, and this fact implies the existence of parametrizations as in (2).

We will prove our results by following Lie's original approach rather closely. The underlying philosophy of the proof is that the integrability condition of the (overdetermined) system of PDE whose solutions are the generalized double translation hypersurfaces with generating curves and surfaces having prescribed tangent directions may be expressed geometrically. The resulting constraints on the generating curves and surfaces of S lead directly to the desired result.

Our major tools will be general facts about *congruences* of curves—analytic two-parameter families of curves in $\mathbb{P}^3$ such as the family of projectivized tangent spaces to one of the generating surfaces of S.

Of course, our major interest in these generalized double translation manifolds comes from the connection between double translation manifolds and theta-divisors given by the Lie-Wirtinger theorem ([7], [5]). Indeed, it is a direct consequence of this theorem that a principally-polarized abelian variety is a Jacobian if and only if its theta-divisor is a double translation manifold. Hence, our local result that sufficiently general *generalized* double translation manifolds are double translation manifolds, combined with this corollary of the Lie-Wirtinger theorem, yields the following global characterization of four-dimensional Jacobians:

Let (A, Θ) be a principally-polarized abelian variety of dimension four with Θ symmetric and assume that there exist a curve C and a codimension-two subvariety V in A such that $\Theta = C + V$ (sum using the group law in A) and C and V are not symmetric themselves. Then under an additional mild hypothesis (see §6) C is a nonhyperelliptic curve of genus four and A is the canonically polarized Jacobian of C.

(If the hypothesis that C and V are not symmetric is removed, then it should still be true that A is a Jacobian, but possibly the Jacobian of a hyperelliptic curve. In the hyperelliptic case, the two parametrizations of Θ as a translation manifold given by Riemann's theorem (that Θ is a translate of W_{g-1} in a Jacobian) coincide.)

Thus, four-dimensional Jacobian varieties are the only principally-polarized abelian varieties of dimension four whose theta-divisors contain even *one* large (two-parameter) family of parallel curves—the translates of the curve C by the points of V. These results may be generalized to give a similar characterization of Jacobians of curves of genus g for all $g \geqslant 4$. The higher-dimensional version of our local result is somewhat more complicated, however, and we will not consider those cases here.

The present paper is organized as follows. §2 contains some preliminaries on the (once-standard) topic of congruences of curves in $\mathbb{P}^3$ (see [1]) and other topics. In §§3 and 4 we study generalized double translation manifolds. In §5 we prove our main local result by applying the techniques of §2 to several congruences of curves which arise naturally from the configuration of projectivized tangent spaces to the generating curves and surfaces of a generalized double translation manifold. Finally, §6 contains the general results on Jacobians obtained from the local theorem of §5.

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2. Some preliminaries

We begin by recalling some of the basic facts about congruences of curves in $\mathbb{P}^3$ or $\mathbb{C}^3$.

(2.1) Definition. A *congruence of curves* is an analytic two-parameter family of analytic curves in an open subset $U \subset \mathbb{P}^3$. The pairs of parameter values may be taken to lie in an open connected subset $V \subset \mathbb{C}^2$.

In the congruences of importance to us, the curves of the congruence will be algebraic curves—either lines or conics.

In naive terms, if the curves of the congruence are defined (perhaps only locally) by equations

$$(1) \qquad f(x_1, x_2, x_3, t_1, t_2), \qquad g(x_1, x_2, x_3, t_1, t_2) = 0$$

(using affine coordinates in U), then the disjoint union of the curves in the congruence can be realized as an analytic subvariety $K \subset U \times V$. If we define $F\colon U \times V \to \mathbb{C}^2$ by $F = (f, g)$, then we take $K = F^{-1}(0,0)$. We will usually assume that $\dim K = 3$ (that is, there are ∞^2 distinct curves in the congruence).

To study how the curves making up a congruence "fit together" in U, we must study the projection π_1:

$$\begin{array}{ccc} & K \subset U \times V & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ U & & V \end{array}$$

(2.2) Definition. A point $p \in K$ is called a *focal point* of K (on the corresponding curve) rank $d\pi_1 \leqslant 2$ at p.

For instance, the focal points of the congruence defined in (1) may be found as follows.

(2.3) Proposition. *The focal points of the congruence* (1) *are the common solutions of* (1) *and*

$$\det\begin{bmatrix} \partial f/\partial t_1 & \partial f/\partial t_2 \\ \partial g/\partial t_1 & \partial g/\partial t_2 \end{bmatrix} = 0.$$

Proof. This follows easily from a calculation of $d\pi_1$ using the chain rule.

(2.4) Examples. If the curves of the congruence K are algebraic, equations (1) and (2) usually yield a *finite* number of focal points on each curve of the congruence. For instance, in a congruence of *lines* defined by equations

$$x_2 = a(t_1, t_2)x_1 + b(t_1, t_2), \qquad x_3 = c(t_1, t_2)x_1 + d(t_1, t_2),$$

the focal point equation (2.3) is quadratic in x_1, so there are *two* focal points on each line (possibly coinciding for some of the lines). Similarly, for congruences of smooth algebraic plane curves of degree n, there will be $n(n+1)$ focal points on each curve of the congruence. This follows from a direct calculation, or by applying the following general result.

(2.5) Proposition. *Let $\mathscr{C}$ be a congruence of smooth algebraic curves of degree d and genus g in $\mathbb{P}^3$. On each curve of the congruence either the number of focal points (counting multiplicities) is exactly $2g - 2 + 4d$, or else every point of the curve is a focal point.*

Proof. As before let $\mathscr{C} \subset U \times V$ be the total space of the family of curves and consider the projection $\pi_1\colon \mathscr{C} \to \mathbb{P}^3$. If it is not the case that every point of every curve of $\mathscr{C}$ is focal, then π_1 is a generically finite mapping. In this case the canonical bundle formula implies that

$$K_{\mathscr{C}} = \pi_1^* K_{\mathbb{P}^3} + R,$$

where R is the ramification divisor of π_1. Of course, by definition, R is just the locus of focal points. Hence if C is a curve of the congruence (viewed as a subvariety of $\mathscr{C}$) and C is not entirely contained in R, then from the formula for $K_{\mathscr{C}}$ and the fact that $K_{\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(-4)$, we see that C meets R in exactly $\deg(K_C) + 4d = 2g - 2 + 4d$ points as claimed. q.e.d.

A useful alternate characterization of the focal points of a congruence K is given by the following proposition.

(2.6) Proposition. *Let K be a congruence and let L be any one-parameter subfamily of K containing a fixed curve C of K. At the focal points of K on C, the tangent plane to the surface swept out by L is the same plane for all such L. Conversely, any point where this is true is a focal point of K.*

Proof. The plane is the image of $d\pi_1$ at that point. q.e.d.

Hence, at each focal point q of K on C, we have a distinguished plane $H_q = \operatorname{Im}(d\pi_1)$ which we will call the *focal plane* of K at q. Note that H_q contains the tangent line to C at q, by the proposition.

We will also need to consider the locus of *all* focal points of the congruence.

(2.7) Definition. The *focal set* of a congruence K, denoted $\mathscr{F}(K)$, is the set of all points $q \in U$ which are focal points on some curve of K passing through q.

In other words, $\mathscr{F}(K)$ is the *branch locus* of the mapping π_1.

As noted before, for a general congruence K, there are only finitely many focal points on each curve of the family, so $\mathscr{F}(K)$ will be, in general, an analytic *surface* in U. One or more of the components or "nappes" of the focal set may degenerate to curves or points for special congruences, however. In any case, we have:

(2.8) Proposition. *Let C be a general curve of a congruence K. At every focal point q of K on C, the focal set $\mathscr{F}(K)$ is tangent to the focal hyperplane H_q. (That is, the tangent space to $\mathscr{F}(K)$ is contained in this hyperplane.)*

Proof. This follows from the observation made before that H_q is the image of $d\pi_1$ at the focal point. q.e.d.

We will be especially interested in congruences of lines in $\mathbb{P}^3$ arising as the family of projectivized tangent spaces to a (two-dimensional) surfaces $A \subset \mathbb{C}^4$. Let K be such a congruence (N. B. not every congruence of lines arises in this way). As might be expected, if $\dim \mathscr{F}(K) < 2$ in this case, then this should reflect some special property of A itself. (By way of analogy, if both nappes of the focal set of the congruence of *normal lines* to a surface in $\mathbb{R}^3$ reduce to curves, then the surface is (part of) a cyclide of Dupin, see [1].)

(2.9) Proposition (*compare* [4, *p.* 414]). *Let $A \subset \mathbb{C}^4$ be a surface which is contained in no three-dimensional affine linear subspace of $\mathbb{C}^4$. Let K be the congruence of projectivized tangent planes to A (a congruence of lines in $\mathbb{P}^3$).*

Assume there are two distinct focal points on each line of K. Then $\dim \mathscr{F}(K) = 1$ *if and only if A is a translation surface (that is, A has a parametrization of the form* $x_i = \gamma_{1i}(s_1) + \gamma_{2i}(s_2)$, $i = 1, \cdots, 4$).

Proof. If A has such a parametrization, then for each $p \in A$ the projectivized tangent plane to A at p is spanned by the points corresponding to the tangent directions of the translates of the generating curves $\gamma_j = (\gamma_{j1}(s_j), \cdots, \gamma_{j4}(s_j))$ passing through p. Thus the lines of the congruence are spanned by pairs of points, one on the curve $\dot{\gamma}_1 = (\gamma'_{11}, \cdots, \gamma'_{14})$, the other on $\dot{\gamma}_2 = (\gamma'_{21}, \cdots, \gamma'_{24})$ in $U \subset \mathbb{P}^3$. Hence, if we fix a point q on $\dot{\gamma}_i$, there are ∞^1 lines of the congruence passing through q. By an easy consequence of Definition (2.1), q is a focal point of K on each line of the congruence which contains it. It follows that $\mathscr{F}(K) = \dot{\gamma}_1 \cup \dot{\gamma}_2$. (Otherwise, there would be at least three focal points on each line of K, so by Proposition (2.5) every point of each of the lines of K would be a focal point. In this case, $\mathscr{F}(K)$ would be a *plane* in $\mathbb{P}^3$, containing all the lines of the congruence, so A would have to be contained in some $\mathbb{C}^3 \subset \mathbb{C}^4$.)

Conversely, if $\dim \mathscr{F}(K) = 1$, by restricting if necessary, we may assume that $\mathscr{F}(K)$ consists of two disjoint analytic curves: $\mathscr{F}(K) = \delta_1 \cup \delta_2$. If we fix a point q on either δ_k, then the lines spanned by q and a variable point on the other δ_i are the projectivized tangent spaces to A at the points of an analytic curve. The curves we obtain for different choices of q are parallel in $\mathbb{C}^4$ (i.e. they are translates of each other). Hence A contains two ∞^1 families of parallel curves. It follows that A is a translation surface. q.e.d.

For future reference, we also include an analysis of the special case in which the two focal points *coincide* on each line of the congruence of projectivized tangent spaces to a surface $A \subset \mathbb{C}^4$. We begin by relating this property to the behavior of the *second fundamental form* $|\mathrm{II}|$ of A, as defined by Griffiths and Harris in [2]. We recall that in this case, for each $p \in A$, $|\mathrm{II}|$ is a pencil of quadrics on $\mathbb{P}T_p(A) \cong \mathbb{P}^1$.

(2.10) Lemma. *Let* $A \subset \mathbb{C}^4$ *be a surface which is not contained in any* $\mathbb{C}^3 \subset \mathbb{C}^4$ *and let K be its associated congruence of projectivized tangent spaces. The two focal points of K coincide on each line if and only if* $|\mathrm{II}|$ *has a base point for all* $p \in A$.

Proof. Let $\mathrm{II} = \mathrm{Span}\{q_1, q_2\}$. Since II may be identified as the differential of the Gauss map on A (see [2, pp. 378–379]), it follows that if Q_1 and Q_2 are the bilinear forms on $T_p(A) \times T_p(A)$ associated to q_1 and q_2 respectively, then the direction of v $(\neq 0) \in T_p(A)$ defines a focal point of K if there is some $w \neq 0$ such that

$$Q_1(v, w) = Q_2(v, w) = 0.$$

By the *symmetry* of the Q_i this implies that

$$Q_1(w, v) = Q_2(w, v) = 0$$

as well, so in fact the direction of w defines the other focal point of K on $\mathbb{P}T_p(A)$. If $w = v$, then by definition $q_1(v) = q_2(v) = 0$, so $|\mathrm{II}|$ has a base point, and conversely. q.e.d.

We will now apply the calculus of moving frames to determine the structure of surfaces $A \subset \mathbb{C}^4$ such that $|\mathrm{II}|$ has a base point for all $p \in A$. The following result is conjectured in [2, p. 377].

(2.11) Proposition. *Let $A \subset \mathbb{C}^4$ be a surface that $|\mathrm{II}|$ has a base point for all $p \in A$. Then either A is a ruled surface, or A lies in a $\mathbb{C}^3 \subset \mathbb{C}^4$ (in which case* II *reduces to a single quadric).*

Proof. We may assume A is contained in no three-dimensional affine linear subspace of $\mathbb{C}^4$. To prove that A is ruled, we will begin by choosing a *special* (Euclidean) *Darboux frame* for A. (We use Euclidean frames rather than projective frames, but otherwise this computation is very similar to many in [2].) This means that we want to find a "good" frame $\{z; e_1, e_2; e_3, e_4\}$ such that z is the position vector of $p \in A$, e_1 and e_2 span $T_p(A)$ (translated to 0), and $\mathbb{C}^4 = T_p(A) + \mathrm{Span}\{e_3, e_4\}$. (The vectors e_3 and e_4 may be thought of as spanning the "normal space" to A at p, but we will *not* use any metric properties of $\mathbb{C}^4$, so this will be merely a complementary subspace to the tangent plane, not the orthogonal complement.) Our special frame will be chosen as follows:

(a) For each p, let e_2 be a vector in the direction of the base point of $|\mathrm{II}|$. (This will correspond to $\omega_1 = 0$ in suitably chosen coordinates.)

(b) In the pencil of quadrics II at p, there is exactly one with a double root: ω_1^2. Choose e_3 so that the corresponding projection of A has second fundamental form ω_1^2 at p.

(c) Choose a *constant* vector e_4 such that $T_p(A) + \mathrm{Span}\{e_3, e_4\}$ equals $\mathbb{C}^4$ for all p and such that the corresponding projection of A has second fundamental form with two distinct roots for each p, say $\mathrm{II} = \omega_1\omega_2$ $(\omega_2 \neq \omega_1)$. (This is clearly true generically for any constant vector e_4, hence we may find such a vector for all p by restricting A if necessary.)

(d) Let $e_1 \in T_p(A)$ be any vector in the direction $\omega_2 = 0$. Hence $\mathrm{II} = \mathrm{Span}\{\omega_1^2, \omega_1\omega_2\}$.

Now the usual structure equations for moving frames yield:

$$dz = \omega_1 e_1 + \omega_2 e_2, \qquad de_i = \sum_{j=1}^{4} \omega_{ij} e_j,$$

$$d\omega_i = \sum_{j=1}^{4} \omega_j \wedge \omega_{ji}, \qquad d\omega_{ij} = \sum_{k=1}^{4} \omega_{ik} \wedge \omega_{kj}. \tag{2}$$

According to the definition of II in [2], if $\sum q_{\alpha\beta\mu}\omega_\alpha\omega_\beta \in$ II corresponds to the "normal direction" e_μ $(3 \leqslant \mu \leqslant 4)$, then

$$\omega_{\alpha\mu} = \sum_\beta q_{\alpha\beta\mu}\omega_\beta.$$

Hence from our choice of frame and the resulting form of II, we obtain

$$\begin{aligned} \omega_{13} &= \omega_1, & \omega_{23} &= 0, \\ \omega_{14} &= \omega_2, & \omega_{24} &= \omega_1. \end{aligned} \tag{3}$$

To show that A is ruled, we will show that the curves $\omega_1 = 0$ on A are straight lines. First, from (2) and (3)

$$\begin{aligned} 0 = d\omega_{23} &= \omega_{21} \wedge \omega_{13} + \omega_{22} \wedge \omega_{23} + \omega_{23} \wedge \omega_{33} + \omega_{24} \wedge \omega_{43} \\ &= \omega_{21} \wedge \omega_1 \end{aligned}$$

since $\omega_{23} = 0$ and $\omega_{43} = 0$ (recall e_4 is constant). From this we deduce that ω_{21} must be a multiple of ω_1.

Now, along the curves $\omega_1 = 0$, we have

$$dz = \omega_1 e_1 + \omega_2 e_2 = \omega_2 e_2.$$

But along these curves,

$$de_2 = \omega_{21}e_1 + \omega_{22}e_2 + \omega_{23}e_3 + \omega_{24}e_4 = \omega_{22}e_2,$$

since $\omega_{23} = 0$ and $\omega_{21} \equiv \omega_{24} \equiv 0 \pmod{\omega_1}$. It follows immediately that the curves $\omega_1 = 0$ are straight lines. (Note that unlike the case of "metric" Darboux frames, we have not normalized our frame vectors to have constant length, so it is possible that $\omega_{22} \neq 0$.)

(2.12) Remarks. (1) The converse of the proposition is also true (and is much easier to prove). See [2, (2.3)].

(2) The same argument shows that if $A \subset \mathbb{C}^n$ $(n \geqslant 4)$ and $|\text{II}|$ has a base point for every $p \in A$, then A is either ruled or A lies in a three-dimensional affine linear subspace of $\mathbb{C}^n$.

Finally, we recall a classical (but, as the referee points out, forgotten) fact from the polar theory of conics in $\mathbb{P}^2$.

Recall that if $Q(x)$ is a homogeneous quadratic form in three variables (defining a conic $\mathscr{C}$: $Q(x) = 0$ in $\mathbb{P}^2$) and $B(x, y)$ is the associated symmetric bilinear form, then for each point $p \in \mathbb{P}^2$, the line $B(p, x) = 0$ is called the *polar* of p with respect to the conic $\mathscr{C}$. A *self-polar triple* with respect to a conic $\mathscr{C}$ is a set $\{p_1, p_2, p_3\}$ of points in $\mathbb{P}^2$ such that for each $\{i, j, k\} = \{1, 2, 3\}$, the line spanned by p_j and p_k is the polar of p_i with respect to $\mathscr{C}$.

If we have two different self-polar triples with respect to the same conic $\mathscr{C}$, then something interesting happens.

(2.13) Proposition. *(This is due, I believe, to Steiner.) Let* $\{p_1, p_2, p_3\}$ *and* $\{q_1, q_2, q_3\}$ *be self-polar triples with respect to a smooth conic* $\mathscr{C}$. *Then all six of the points lie on (another) conic* Γ.

Proof. There are many ways to prove this. We will give an algebraic argument. We will assume the six points are distinct.

The key idea from the algebraic standpoint is that, by the definition of a self-polar triple, if we change coordinates and use the p_i or the q_i as the vertices of the reference triangle, then the quadratic form Q defining the conic $\mathscr{C}$ will be taken to *diagonal form* in the new coordinates.

To simplify the computations, we may assume that coordinates have been chosen so that Q has the form:

$$Q(x_0, x_1, x_2) = x_0^2 + x_1^2 + x_2^2$$

(i.e., the vertices of the standard reference triangle in $\mathbb{P}^2$ themselves form a self-polar triple with respect to $\mathscr{C}$).

If we write

$$p_i = (r_{i0}, r_{i1}, r_{i2}) \quad \text{and} \quad q_j = (r_{j+3,0}, r_{j+3,1}, r_{j+3,2}),$$

then the polar of (r_{i0}, r_{i1}, r_{i2}) with respect to $\mathscr{C}$ is the line

$$r_{i0}x_0 + r_{i1}x_1 + r_{i2}x_2 = 0$$

and we have

$$\begin{aligned} Q(x_0, x_1, x_2) &= \sum_{i=1}^{3} \alpha_i (r_{i0}x_0 + r_{i1}x_1 + r_{i2}x_2)^2 \\ &= \sum_{i=4}^{6} (-\alpha_i)(r_{i0}x_0 + r_{i1}x_1 + r_{i2}x_2)^2 \end{aligned}$$

for some $\alpha_i \neq 0$ in $\mathbb{C}$. Hence

$$\sum_{i=1}^{6} \alpha_i (r_{i0}x_0 + r_{i1}x_1 + r_{i2}x_2)^2 \equiv 0.$$

This identical vanishing implies

$$\sum_{i=1}^{6} \alpha_i r_{ik} r_{il} = 0 \tag{4}$$

for all pairs (k, l).

Now, suppose Γ is a conic containing five of the points in the two triples. If Γ has an equation $\sum_{k,l} c_{kl} x_k x_l = 0$, then we may assume without loss of generality that

$$\sum_{k,l} c_{kl} r_{ik} r_{il} = 0 \tag{5}$$

for $i = 1, \cdots, 5$.

On the one hand,

$$\sum_{i=1}^{6} \alpha_i \left(\sum_{k,l} c_{kl} r_{ik} r_{il} \right) = \alpha_6 \cdot \sum_{k,l} c_{kl} r_{6k} r_{6l}$$

by (5). On the other hand interchanging the order of summation on the left, this also equals

$$\sum_{k,l} c_{kl} \cdot \left(\sum_{i=1}^{6} \alpha_i r_{ik} r_{il} \right),$$

which is zero by (4). Hence Γ contains the sixth point as well.

3. Generalized translation manifolds

Our major objects of study will be the hypersurfaces $S \subset \mathbb{C}^{n+1}$ satisfying the condition given in the following definition.

(3.1) Definition. We will call an analytic hypersurface $S \subset \mathbb{C}^{n+1}$ a *generalized translation hypersurface* if S has a parametrization of the form

$$x_i = \alpha_i(t_1) + A_i(t_2, \cdots, t_n) \qquad (1 \leqslant i \leqslant n+1), \tag{1}$$

where the α_i and the A_i are analytic functions.

We will assume that the origin $0 \in S$ and that our parametrizations have been chosen so that

$$\alpha(0) = (\alpha_1(0), \cdots, \alpha_{n+1}(0)) = 0,$$
$$A(0, \cdots, 0) = (A_1(0, \cdots, 0), \cdots, A_{n+1}(0, \cdots, 0)) = 0.$$

A generalized translation hypersurface is swept out by translating the analytic curve $\alpha = \alpha(t_1)$ by the points of the codimension-two subvariety $A = A(t_2, \cdots, t_n)$ in $\mathbb{C}^{n+1}$, or by translating A along α. Note that given any pair α, A we can construct such a hypersurface. We call α and A the *generators* of S.

In this paper, we will specialize to the case $n = 3$, and consider hypersurfaces $S \subset \mathbb{C}^4$ which have a parametrization of the form (1):

$$x_i = \alpha_i(t_1) + A_i(t_2, t_3). \tag{2}$$

The geometry of such hypersurfaces is controlled by the configuration of the tangent spaces to the translates of the generators α and A passing through each $p \in S$. To see the picture more clearly, we follow Lie, projectivize the situation in each tangent space to S, and view everything in one fixed $\mathbb{P}^3 = \mathbb{P}(\mathbb{C}^4)$. (We are identifying all the spaces $T_p(\mathbb{C}^4) = \mathbb{C}^4$ for $p \in S$.)

When this is done, in the projectivized tangent space $H = \mathbb{P}T_p(S)$ (a $\mathbb{P}^2 \subset \mathbb{P}^3$) we have:

(a) a *point*, the projectivized tangent line at p to the translate of the generating curve α passing through p, and

(b) a *line*, the projectivized tangent space at p to the translate of the generating surface A passing through p.

As $p \in S$ varies, in some family of planes in an open set $U \subset \mathbb{P}^3$ we obtain:

(a) a *curve*, the curve of tangent directions to α, which we will call $\dot{\alpha}$, and

(b) a *two-parameter family of lines*, K, the congruence of projectivized tangent spaces to $A \subset \mathbb{C}^4$ as in §2.

Our first goal is to identify and rule out some relatively uninteresting *degenerate* generalized translation hypersurfaces which may be recognized by special behavior of the second fundamental forms of S and A. First we note that the second fundamental form $\mathrm{II}(S)$ at $p \in S$ is given by a single conic $F(H) \subset H = \mathbb{P}T_p(S)$. Indeed, if S is defined by an equation of the form

$$x_4 = f(x_1, x_2, x_3),$$

then $F(H)$ is defined by

$$\sum_{i,j} f_{ij}(p)\xi_i\xi_j = 0,$$

where $f_{ij} = \partial^2 f/\partial x_i \partial x_j$. Similarly, $\mathrm{II}(A)$ is spanned by two quadrics. It is easily checked that $\mathrm{II}(S)|_{T_p(A)} \in \mathrm{II}(A)$.

One way to catalogue some of the possibilities for degenerate S is to recall the relation between the rank of the conic $F(H)$ and the dimension of the image of the (projectivized) *Gauss map* on S:

$$\begin{aligned} \gamma\colon S &\to G(2,3) \qquad \text{(the Grassmanian of planes in } \mathbb{P}^3\text{)} \\ p &\to \mathbb{P}T_p(S). \end{aligned}$$

By [2, (2.6)] we have

$\dim \mathrm{Im}(\gamma) = 3$ if and only if $F(H)$ is smooth for general p,

$\dim \mathrm{Im}(\gamma) = 2$ if and only if $F(H)$ consists of a pair of distinct lines for general p,

$\dim \mathrm{Im}(\gamma) = 1$ if and only if $F(H)$ is doubled line for general p, and

$\dim \mathrm{Im}(\gamma) = 0$ if and only if $\mathrm{II}(S) = 0$ everywhere.

If $\dim \mathrm{Im}(\gamma) \leqslant 2$ we will say that S is *developable*, by analogy with the classical terminology for surfaces. In each case we can analyze the structure of a generalized translation manifold with a degenerate Gauss map, as follows.

(3.2) If $\dim \mathrm{Im}(\gamma) = 0$, then S is a *hyperplane* in $\mathbb{C}^4$. Every hyperplane is a generalized translation manifold in *infinitely many ways*.

(3.3) If $\dim \operatorname{Im}(\gamma) = 1$, then S is swept out by ∞^1 2-planes in $\mathbb{C}^4$. There are several different types of generalized translation manifolds with $\dim \operatorname{Im}(\gamma) = 1$. If the generating curve α is not a straight line, then for $\dim \operatorname{Im}(\gamma) = 1$ we must have that A is a 2-plane. Alternately, if α is a line, then S contains the ∞^2 lines obtained by translating α by the points of A. Hypersurfaces S with $\dim \operatorname{Im}(\gamma) = 1$ are obtained, for example, if A is a developable ruled surface (or a cone). These hypersurfaces are also generalized translation manifolds in infinitely many ways.

(3.4) If $\dim \operatorname{Im}(\gamma) = 2$, then S is swept out by ∞^2 lines in $\mathbb{C}^4$. Such surfaces are obtained, for example, by translating any surface A along a line α, or by translating a developable ruled surface or cone along any curve α. In any case, S is the union of ∞^1 developable ruled surfaces (or cones) in more than one way.

(3.5) Finally, we note that if $|\mathrm{II}(A)|$ has a base point for all $p \in A$, then by Proposition (2.11), A is either contained in a $\mathbb{C}^3$ or A is ruled. In the second case, S is swept out by the ∞^2 lines obtained by translating the lines of the ruling on A along the curve α.

From this discussion, we conclude that if S does not contain ∞^2 lines (that is S is neither developable nor contains one of the special ruled surfaces A discussed in (3.5)), then by restricting S if necessary, we may assume that:

(a) There is a one-to-one correspondence between planes H near $H^0 = \mathbb{P}T_0(S)$ and points p near 0 in S given by $p \leftrightarrow \mathbb{P}T_p(S)$. In each such plane we have a point $\dot{\alpha} \cap H$ from the curve $\dot{\alpha}$ and a line $k(H)$ from the congruence K of projectivized tangent spaces to A.

(b) The curve $\dot{\alpha}$ crosses each H transversely.

(c) On each line $k(H)$ there are two *distinct* focal points of K, which we will denote by $P_1(H)$ and $P_2(H)$.

In this situation, we have the following relationship between the points $\{\dot{\alpha} \cap H, P_1(H), P_2(H)\}$ for each H near H^0.

(3.6) Proposition. $\{\dot{\alpha} \cap H, P_1(H), P_2(H)\}$ *is a self-polar triple (see* §2) *with respect to the smooth conic* $F(H)$—*the second fundamental form of* S *at the point* p *with* $H = \mathbb{P}T_p(S)$.

Proof. We may assume that S has a local equation of the form

$$x_4 = f(x_1, x_2, x_3).$$

From the parametrization

$$x_i = \alpha_i(t_1) + A_i(t_2, t_3),$$

if we substitute into the equation of S and compute mixed partial derivatives, we see that

$$0 = \sum_{i,j} f_{ij} \cdot \alpha_i'(t_1) \cdot \frac{\partial A_j}{\partial t_2}, \qquad 0 = \sum_{i,j} f_{ij} \cdot \alpha_i'(t_1) \cdot \frac{\partial A_j}{\partial t_3},$$

where the prime denotes differentiation with respect to t_1 and as before we write f_{ij} for $\partial^2 f/\partial x_i \partial x_j$.

From these equations we see immediately that the line $k(H)$ (spanned by the tangent directions $\partial/\partial t_2$ and $\partial/\partial t_3$) is the *polar* of the point $\dot{\alpha} \cap H$ with respect to $F(H)$.

Next, from the facts used in the proof of Lemma (2.10), we see that since the quadric $\mathrm{II}(S)|_{T(A)}$ is in the pencil $\mathrm{II}(A)$, the tangent directions corresponding to the focal points of K on $k(H)$ are *conjugate* with respect to $\mathrm{II}(S)$. The claim follows immediately.

(3.7) Remark. This result holds even in the case where $A \subset \mathbb{C}^3 \subset \mathbb{C}^4$. In that case, every point on each of the lines $k(H)$ is a focal point of the line congruence K. Given a point $P_1(H)$, unless the single quadric in $\mathrm{II}(A)$ is a square at every $p \in A$ (which implies that A is a developable ruled surface or cone), there will be another uniquely determined point $P_2(H)$ such that $\{\dot{\alpha} \cap H, P_1(H), P_2(H)\}$ is a self-polar triple with respect to $F(H)$. The point $P_2(H)$ corresponds to the conjugate direction to $P_1(H)$ on A (with respect to $\mathrm{II}(A)$).

4. Generalized double translation hypersurfaces

We now turn to hypersurfaces S having *two* parametrizations as a generalized translation manifold:

(1) $$x_i = \alpha_i(t_1) + A_i(t_2, t_3) = \beta_i(u_1) + B_i(u_2, u_3).$$

It is important to note that although the generating curves and surfaces of a generalized (single) translation hypersurface are completely arbitrary, the same is definitely *not* true for a generalized double translation hypersurface. Analytically, this is a reflection of the fact that the system of PDE whose solutions are these hypersurfaces is overdetermined. In this section, we will derive a striking geometric form of the integrability conditions of this system, following Lie's ideas.

By the results of §3, we may assume that S is not one of the developable hypersurfaces considered in (3.2)–(3.4) and S is not one of the ruled three-folds considered in (3.5). In this case, the hypotheses of Proposition (3.6) will apply to both the triple $\{\dot{\alpha} \cap H, P_1(H), P_2(H)\}$ from the first parametrization of S and the analogous triple $\{\dot{\beta} \cap H, Q_1(H), Q_2(H)\}$ from the second. (Here $Q_i(H)$ are the two focal points on the line $l(H)$ from the congruence L of projectivized tangent spaces to the generating surface B.) By Proposition (3.6) *both* of these triples are self-polar triples with respect to the conic $F(H)$.

Our main interest is in the case in which the two triples are in general position in H (see Figure 1).

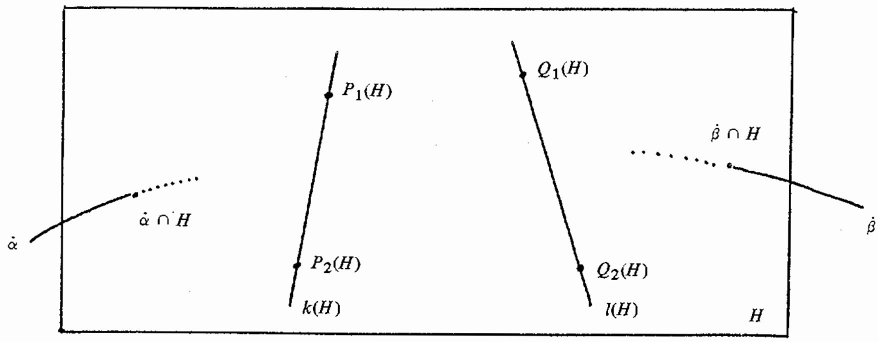


FIGURE 1. The general situation

However, there are several ways this can fail to be true, yielding some degenerate special types of generalized double translation hypersurfaces. We will now identify the hypersurfaces obtained in each case.

(4.1) One of the points of the first triple coincides with one of the points of the second triple (for all H).

In this case, the line spanned by the other two points of the first triple must coincide with the line spanned by the remaining two points of the second triple as well. This is true since that line is the polar of the coincident points with respect to $F(H)$. There are now several subcases which arise depending on whether:

(a) $\dot\alpha \cap H$ coincides with $\dot\beta \cap H$, for all H, or

(b) (relabeling if necessary) $\dot\alpha \cap H$ coincides with one of the $Q_i(H)$ for all H, or

(c) one of the $P_i(H)$ coincides with one of the $Q_j(H)$ for all H.

In case (a), we may characterize the situation quite simply by noting that this happens if and only if the generating curves α and β *coincide*. The two parametrizations in (1) are therefore *not really distinct*. Examples of such hypersurfaces may be obtained by taking any surface (described by two possibly different parametrizations) and translating along any curve. A (somewhat more interesting) special case arises if the surface A is a *double translation surface* (which then necessarily lies in a $\mathbf{C}^3$, since there are more than two focal points on each line of the associated congruence of projectivized tangent spaces) and S is generated by translating A along the curve $\alpha = \beta$. This gives a special degenerate double translation manifold. However, these cases should clearly be excluded from our further considerations, since we want to consider generalized double translation hypersurfaces with two distinct parametrizations.

Cases (b) and (c) are best seen as further degenerations of cases to be considered later, so we will not discuss them now.

Modulo this, let us continue and assume that the six points of our two self-polar triples are *distinct*. The next case we will consider is:

(4.2) The six points are distinct but one of the lines spanned by a pair from the first triple coincides with one of the lines spanned by a pair from the second (see Figure 2).

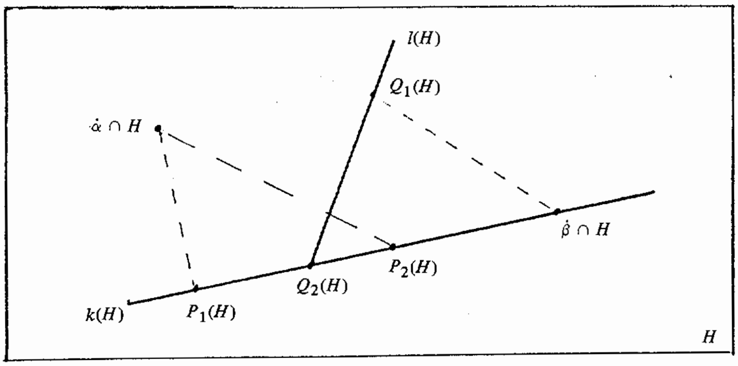


FIGURE 2.

This case may be excluded immediately, since if this occurs for all H (and there are no coincident points), then S must in fact be *developable*. The reason is that if not, as we have seen, the conic $F(H)$ defined by $\mathrm{II}(S)$ is smooth for general H. If this is true then the polarity mapping $\pi\colon \mathbb{P}^2 \to (\mathbb{P}^2)^*$ defined by $\pi(p) =$ polar of p with respect to $F(H)$ is *injective*.

Hence, we are now reduced to considering the following situation:

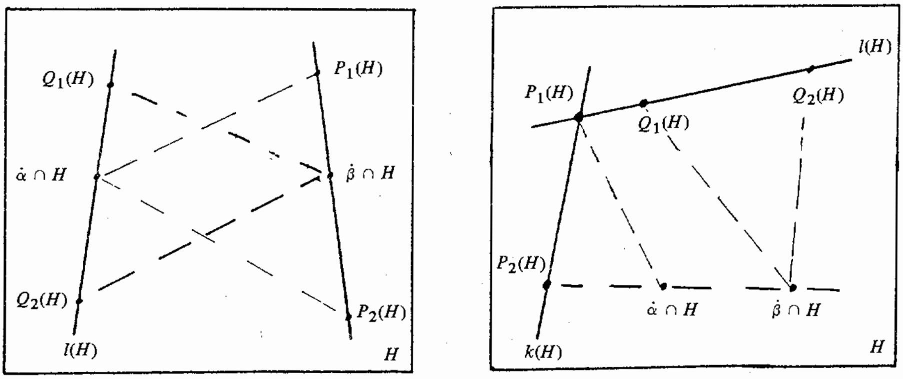
(4.3) The triangles defined by the two self-polar triangles have no vertices or edges in common, but a vertex of one is contained in an edge of the other for all H.

To analyze this case, we will use the following direct consequence of Proposition (2.13).

(4.4) Proposition. *In the case that the points of the two self-polar triples are distinct, there is a unique conic* $\Gamma(H)$ *containing* $\{\dot{\alpha} \cap H, P_1(H), P_2(H)\} \cup \{\dot{\beta} \cap H, Q_1(H), Q_2(H)\}$.

It is the existence of the conic $\Gamma(H)$ that makes our configuration of generating curves and surfaces in S really special. This is the geometric form of the integrability conditions mentioned before.

If we are in the situation (4.3) then it is clear that the conic $\Gamma(H)$ must split into two lines, since it contains three collinear points. Since the two triangles are assumed to have no edges in common, the remaining three points are *also collinear*. Without loss of generality, we may reduce to studying the following two cases (see Figure 3).



Situation (4.5a) Situation (4.5b)

FIGURE 3.

(4.5a) $\dot\alpha \cap H$ lies in $l(H) = \text{Span}\{Q_1(H), Q_2(H)\}$ and $\dot\beta \cap H$ lies in $k(H) = \text{Span}\{P_1(H), P_2(H)\}$ for all H.

Since A contains ∞^1 translates of $\dot\beta$ and similarly B contains ∞^1 translates of $\dot\alpha$ it follows that both A and B are themselves translation surfaces and that S is a degenerate type of double translation hypersurface. We note that subcase (4.1(b)) may be seen as a specialization of this one, in which $\dot\alpha \cap H$ actually coincides with one of the $Q_i(H)$ for all H. Hence those hypersurfaces are also double translation hypersurfaces for the same reason.

(4.5) One of the $P_i(H)$, say $P_1(H)$, lies in $l(H) = \text{Span}\{Q_1(H), Q_2(H)\}$ for all H.

Here by fixing a point on $\dot\alpha$ and letting the point on $\dot\beta$ vary, it may be seen that B (and hence A as well) are translation surfaces, since the curves on B and A obtained by integrating the field of tangent directions given by $P_1(H)$ must all be translates of each other. As a result, S is a double translation manifold of a degenerate type. Case (4.1(c)) is a further degeneration of this one.

Having disposed of these degenerate cases, we are now ready to tackle the remaining (and most interesting) case in which the points of the two triples are in general position for general H, and consequently the conic $\Gamma(H)$ is *smooth* for general H.

5. The local theorem

We are now ready to state and prove our main local result about generalized double translation hypersurfaces in $\mathbb{C}^4$. Our method will be to use the properties of the family of conics $\Gamma(H)$ deduced in §4 to show that in the case that the generating curves and surfaces of S are in general position it must be the case that both $\mathcal{F}(K)$ are one-dimensional. Once we have this, we will then apply Proposition (2.8) to obtain our desired conclusion. Lie also knew of the existence of the conic $\Gamma(H)$ (see [3, p. 435]), but it apparently did not occur to him to use the behavior of all the conics in the family to obtain further information on the focal sets $\mathcal{F}(K)$ and $\mathcal{F}(L)$. (Note that $\mathcal{F}(K)$ and $\mathcal{F}(L)$ are contained in the union of the $\Gamma(H)$.)

(5.1) Theorem. *Let $S \subset \mathbb{C}^4$ be a generalized double translation hypersurface. Assume that neither of the generating surfaces A and B is contained in a $\mathbb{C}^3 \subset \mathbb{C}^4$. Then either*

(a) *S is developable, or*

(b) *S is ruled by ∞^2 lines in $\mathbb{C}^4$, or*

(c) *the generating surfaces A and B are translation surfaces so that S is a double translation hypersurface.*

Proof. By our analysis of the degenerate cases (3.2)–(3.5), (4.1)–(4.3), and (4.5) the only case left to consider is the one in which the two self-polar triples $\{\dot{\alpha} \cap H, P_1(H), P_2(H)\}$ and $\{\dot{\beta} \cap H, Q_1(H), Q_2(H)\}$ lie on a smooth conic $\Gamma(H)$ in all planes H near $H^0 = \mathbb{P}T_0(S)$.

Let $C(p_0)$ be the congruence of *conics* $\Gamma(H)$ lying in the planes H containing $p_0 = \dot{\alpha} \cap H^0$. Since p_0 is contained in each of these conics, it follows from the definition that p_0 counts as a *double* focal point on each of them.

I claim that the focal points $P_i(H)$ and $Q_i(H)$ of the line congruences K and L are *also* focal points of $C(p_0)$ on $\Gamma(H)$. To see this, consider the total space Γ of the whole family of conics $\Gamma(H)$, and the mapping π_2, whose image is an open set in $(\mathbb{P}^3)^*$ (the dual projective space). Since we assume S is not developable, there is also a one-to-one correspondence between the points of this set and the points of S near 0.

$$\begin{array}{ccc} & \Gamma & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{P}^3 \supset U & & V \subset (\mathbb{P}^3)^* \end{array}$$

We have $\dim \Gamma = 4$ and at any point on Γ we can use as local coordinates t_1, t_2, t_3 on S and any local parameter on the conic $\Gamma(H)$. Consider the point $P_1(H)$, for example. Clearly, the image of $P_1(H)$ under π_1 is unchanged in the

t_1 direction on S. Furthermore, since $P_1(H)$ is a focal point of the line congruence K associated to $A \subset S$, it follows that the derivative of π_1 vanishes in one direction in the $(\partial/\partial t_2, \partial/\partial t_3)$ plane as well as at $P_1(H)$. Hence rank $d\pi_1 \leqslant 2$ at $P_1(H)$. When we restrict to the two-parameter subfamily $C(p_0)$, the rank cannot increase, and by Definition (2.2) $P_1(H)$ is a focal point of $C(p_0)$ on $\Gamma(H)$. The same is true for the other points $P_2(H)$, $Q_1(H)$, $Q_2(H)$.

Finally, by the definition of focal points of congruences, it follows that the point $\dot{\beta} \cap H$ is *also* a focal point on each of the conics in $C(p_0)$. Since every conic in the two-parameter family meets the curve $\dot{\beta}$ there is a one-parameter subfamily containing each of the points on $\dot{\beta}$ and the intersection points are focal points.

Since each conic $\Gamma(H)$ in the congruence $C(p_0)$ has at least *seven* focal points (counting multiplicities), by Proposition (2.5) every point of every conic in the congruence $C(p_0)$ is a focal point. Hence, in the notation of §2, if π_1: $C(p_0) \to \mathbb{P}^3$ is the projection, we have rank $d\pi_1 = 2$ at every point. It follows that all the conics of the congruence lie on a two-dimensional surface $M \subset U \subset \mathbb{P}^3$.

Since the focal points of the line congruences K and L lie on the conics of $C(p_0)$, this implies that the focal sets $\mathscr{F}(K)$ and $\mathscr{F}(L)$ are *contained* in the surface M.

Now recall that by Proposition (2.7) the focal sets $\mathscr{F}(K)$ and $\mathscr{F}(L)$ are *tangent* to the focal plane of the corresponding line congruence at each focal point. However, this focal hyperplane contains the line $k(H)$ (resp. $l(H)$). Since the lines $k(H)$ and $l(H)$ meet the conic $\Gamma(H)$ transversely, it follows that $\mathscr{F}(K)$ and $\mathscr{F}(L)$ must reduce to *curves* lying on the surface M. By Proposition (2.8), then, the surfaces A and B in S must be translation surfaces, with generating curves given by integrating the fields of tangent directions given by the focal points.

(5.2) Remarks. It also follows from what we have seen already that *all* the conics $\Gamma(H)$, not only the ones in the congruence $C(p_0)$, lie on a fixed surface $M \subset \mathbb{P}^3$, and hence that this surface is an open subset of a quadric surface. The curves $\dot{\alpha}$, $\dot{\beta}$, and the components of $\mathscr{F}(K)$ and $\mathscr{F}(L)$ are curves lying on this quadric surface.

With the advantage of hindsight (see §6) it may be seen that these six (a priori analytic) curves on the quadric are actually parts of one algebraic curve of degree six lying on the quadric—a canonical curve of genus four or a singular limit of such curves. It should be possible to prove this directly, perhaps by showing that the "Reiss-type" relations which characterize the algebraic curves lying on a quadric surface (see [6, pp. 79–80]) are satisfied here. This would give a self-contained proof of the results of §6 of this paper by methods very close to Lie's original approach to the problem. The author has not been able to carry this out as of yet, however.

6. Some applications

In this section we will show how the local result of the previous section may be combined with the Lie-Wirtinger theorem on double translation manifolds ([5], [7]) to obtain another characterization of Jacobians of nonhyperelliptic curves of genus four. Our first result will be a direct application of the corollary of the Lie-Wirtinger theorem given in [5, Theorem (5.1)].

(6.1) Theorem. *Let (A, Θ) be a principally polarized abelian variety of dimension four with Θ irreducible and assume that in a neighborhood of $p \in \Theta$, Θ (or its lift to $\mathbf{C}^4$, suitably translated) has two distinct parametrizations of the form.*

$$x_i = \alpha_i(t_1) + A_i(t_2, t_3) = \beta_i(u_1) + B_i(u_2, u_3).$$

Assume that the parametrizations satisfy the following additional condition:

(*) *If we vary t_1 leaving t_2, t_3 fixed, then all the u_j vary along the curve traced out in Θ.*

Then (A, Θ) is the canonically polarized Jacobian of a nonhyperelliptic curve of genus four.

Proof. First, by Theorem (5.1), since an irreducible theta-divisor in an abelian variety is never developable or ruled, Θ is a double translation manifold with two distinct parametrizations

$$x_i = \alpha_i(t_1) + \alpha_{2i}(\bar{t}_2) + \alpha_{3i}(\bar{t}_3) = \beta_i(u_1) + \beta_{2i}(\bar{u}_2) + \beta_3(\bar{u}_3).$$

Since (*) is satisfied for the original parametrization, by the fact that our generating curves and surfaces are in general position, the analogous condition is satisfied for $\bar{u}_2$ and $\bar{u}_3$ as well. Hence Theorem (3.9) of [5] applies and the claim follows. q.e.d.

We can obtain another sort of characterization of Jacobians if we proceed as follows. Again, let (A, Θ) be a four-dimensional principally polarized abelian variety and suppose that Θ is a symmetric theta-divisor. Suppose that Θ may be generated globally by translating a curve $C \subset A$ along a two-dimensional surface $V \subset A$.

(6.2) Corollary. *Assume that neither C nor V is symmetric and that the second parametrization of Θ as a generalized translation manifold obtained by reflecting the given one through the origin in A satisfies the hypotheses of Theorem* (6.1). *Then C is a curve genus four and (A, Θ) is the canonically polarized Jacobian of C.*

(6.3) Remark. It would be interesting to see how (or whether) the other approaches to characterizing Jacobian varieties (in particular the Schottky relation) are related to this geometric characterization of Jacobians. Can they be interpreted as guaranteeing the existence of two-parameter family of "parallel" curves in Θ (the translates of C by the points of V)?

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